

CAD ADJACENCY COMPUTATION USING VALIDATED NUMERICS

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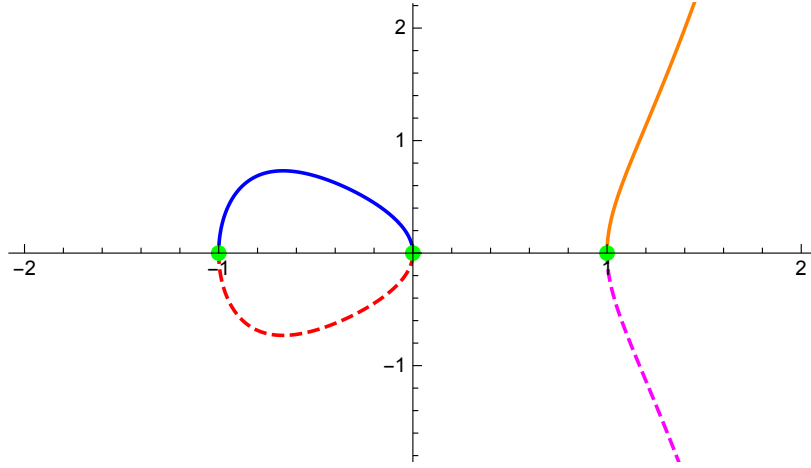
ABSTRACT. We present an algorithm for computation of cell adjacencies for well-based cylindrical algebraic decomposition. Cell adjacency information can be used to compute topological operations e.g. closure, boundary, connected components, and topological properties e.g. homology groups. Other applications include visualization and path planning. Our algorithm determines cell adjacency information using validated numerical methods similar to those used in CAD construction, thus computing CAD with adjacency information in time comparable to that of computing CAD without adjacency information. We report on implementation of the algorithm and present empirical data.

1. INTRODUCTION

A semialgebraic set is a subset of \mathbb{R}^n which is a solution set of a system of polynomial equations and inequalities. Computation with semialgebraic sets is one of the core subjects in computer algebra and real algebraic geometry. A variety of algorithms have been developed for real system solving, satisfiability checking, quantifier elimination, optimization and other basic problems concerning semialgebraic sets [7, 3, 5, 9, 10, 11, 13, 19, 24, 25]. Every semialgebraic set can be represented as a finite union of disjoint cells bounded by graphs of semialgebraic functions. The Cylindrical Algebraic Decomposition (CAD) algorithm [7, 5, 21] can be used to compute a cell decomposition of any semialgebraic set presented by a quantified system of polynomial equations and inequalities. Alternative methods of computing cell decompositions are given in [6, 22, 23]. For solving certain problems, for instance computing topological properties or visualization, it is not sufficient to know a cell decomposition of the set, but it is also necessary to know how the cells are connected together.

Example 1. *The CAD algorithm applied to the equation $y^2 = x(x^4 - 1)$ gives four one-dimensional cells and three zero-dimensional cells shown in Figure 1.1. To find the connected components of the solution set it is sufficient to know which one-dimensional cells are adjacent to which zero-dimensional cells.*

Several algorithms for computing cell adjacencies have been developed. The algorithm given in [20] computes cell adjacencies for CAD that are well-based. A CAD is well-based if none of the polynomials whose roots appear in cell description vanishes identically at any point. This is a somewhat stronger condition than well-orientedness required for the McCallum projection [15], nevertheless a large portion of examples that appear in practice satisfies the condition. In a well-based CAD all cell adjacencies can be determined from adjacencies between cells whose dimensions differ by one. In \mathbb{R}^2 all CADs are well-based. Algorithm for computing cell adjacencies in arbitrary CADs in \mathbb{R}^3 has been given in [1]. For determining cell adjacencies [20] proposes methods based on fractional power series representations of polynomial roots. Another method, given in [16], computes adjacencies

FIGURE 1.1. $y^2 = x(x^4 - 1)$

between a zero-dimensional cell and one-dimensional cells by analyzing intersections of the one dimensional cells with sides of a suitable box around the zero-dimensional cell. For an alternative method of computing connectivity properties of semialgebraic sets see [4, 2, 3, 8].

In this paper we present a new algorithm which computes cell adjacencies for well-based CAD. The algorithm uses validated numerical methods similar to those used in [21] for construction of CAD cell sample points. The method is based on computation of approximations of polynomial roots and increasing the precision of computations until validation criteria are satisfied. Unlike the previously known algorithms, it does not require polynomial computations over algebraic number fields or computation with fractional power series representations of polynomial roots. Also, unlike the CAD construction algorithm given in [21], the algorithm never needs to revert to exact algebraic number computations. We have implemented the algorithm as an extension to the CAD implementation in *Mathematica*. Empirical results show that computation of CAD with cell adjacency data takes time comparable to computation of CAD without cell adjacency data.

The general idea of the algorithm is as follows. It starts, similarly as the CAD algorithm, with computing a sample point in each cell in \mathbb{R}^k for all $k \leq n$. The sample point of a cell in \mathbb{R}^{k+1} extends the sample point of the projection of the cell on \mathbb{R}^k . Then for each pair of adjacent CAD cells C and C' in \mathbb{R}^k with $\dim C' = \dim C - 1$ the algorithm constructs a point $p \in C$ that is “sufficiently close” to the sample point p' of C' . Here “sufficiently close” means that computing approximations of roots of projection polynomials at p and p' is sufficient to identify which roots over C tend to which roots over C' and to continue the construction to pairs of adjacent CAD cells in \mathbb{R}^{k+1} . The construction gives all pairs of adjacent cells in \mathbb{R}^n whose dimensions differ by one. For well-based CAD this is sufficient to determine all cell adjacencies.

2. PRELIMINARIES

A system of polynomial equations and inequalities in variables x_1, \dots, x_n is a formula

$$S(x_1, \dots, x_n) = \bigvee_{1 \leq i \leq l} \bigwedge_{1 \leq j \leq m_i} f_{i,j}(x_1, \dots, x_n) \rho_{i,j} 0$$

where $f_{i,j} \in \mathbb{R}[x_1, \dots, x_n]$, and each $\rho_{i,j}$ is one of $<, \leq, \geq, >, =$, or \neq .

A subset of \mathbb{R}^n is *semialgebraic* if it is a solution set of a system of polynomial equations and inequalities.

A *quantified system of real polynomial equations and inequalities* in free variables x_1, \dots, x_m and quantified variables x_{m+1}, \dots, x_n is a formula

$$(2.1) \quad Q_1 x_{m+1} \dots Q_n x_n S(x_1, \dots, x_n)$$

Where Q_i is \exists or \forall , and S is a system of real polynomial equations and inequalities in x_1, \dots, x_n .

By Tarski's theorem (see [24]), solution sets of quantified systems of real polynomial equations and inequalities are semialgebraic.

Notation 2. For $k \geq 1$, let \bar{a} denote a k -tuple (a_1, \dots, a_k) of real numbers and let \bar{x} denote a k -tuple (x_1, \dots, x_k) of variables.

Every semialgebraic set can be represented as a finite union of disjoint *cells* (see [12]), defined recursively as follows.

- (1) A cell in \mathbb{R} is a point or an open interval.
- (2) A cell in \mathbb{R}^{k+1} has one of the two forms

$$\begin{aligned} & \{(\bar{a}, a_{k+1}) : \bar{a} \in C_k \wedge a_{k+1} = r(\bar{a})\} \\ & \{(\bar{a}, a_{k+1}) : \bar{a} \in C_k \wedge r_1(\bar{a}) < a_{k+1} < r_2(\bar{a})\} \end{aligned}$$

where C_k is a cell in \mathbb{R}^k , r is a continuous semialgebraic function, and r_1 and r_2 are continuous semialgebraic functions, $-\infty$, or ∞ , and $r_1 < r_2$ on C_k .

For a cell $C \subseteq \mathbb{R}^n$ let $\Pi_k(C) \subseteq \mathbb{R}^k$, for $k \leq n$, denote the projection of C on \mathbb{R}^k . A finite collection D of cells in \mathbb{R}^n is *cylindrically arranged* if for any $C_1, C_2 \in D$ and any $k \leq n$ $\Pi_k(C_1)$ and $\Pi_k(C_2)$ are either disjoint or identical.

A *cylindrical algebraic decomposition (CAD)* of \mathbb{R}^n is a finite collection D of pairwise disjoint cylindrically arranged cells in \mathbb{R}^n such that $\bigcup_{C \in D} C = \mathbb{R}^n$.

Let $P \subset \mathbb{R}[x_1, \dots, x_n]$ be a finite set of polynomials. A CAD D of \mathbb{R}^n is *P-invariant* if each element of P has a constant sign on each cell of D .

Let $A \subseteq \mathbb{R}^n$ be a semialgebraic set. A CAD D of \mathbb{R}^n is *consistent with A* if $A = \bigcup_{C \in D_A} C$ for some $D_A \subseteq D$.

Let $C_1, C_2 \in D$. C_1 and C_2 are *adjacent* if $C_1 \neq C_2$ and $C_1 \cup C_2$ is connected.

For a semialgebraic set A presented by a quantified system of polynomial equations and inequalities (2.1), the CAD algorithm can be used to find a CAD D of \mathbb{R}^n consistent with A . The CAD D is represented by a cylindrical algebraic formula (CAF). A CAF describes each cell by giving explicit semialgebraic function bounds and the Boolean structure of a CAF reflects the cylindrical arrangement of cells. Before we give a formal definition of a CAF, let us first introduce some terminology.

Let $k \geq 1$ and let $f = c_d y^d + \dots + c_0$, where $c_0, \dots, c_d \in \mathbb{Z}[\bar{x}]$. A *semialgebraic function* given by the *defining polynomial* f and a *root number* $\lambda \in \mathbb{N}_+$ is the function

$$(2.2) \quad \text{Root}_{y,\lambda} f : \mathbb{R}^k \ni \bar{a} \longrightarrow \text{Root}_{y,\lambda} f(\bar{a}) \in \mathbb{R}$$

where $\text{Root}_{y,\lambda}f(\bar{a})$ is the λ -th real root of $f(\bar{a}, y) \in \mathbb{R}[y]$. The function is defined for those values of \bar{a} for which $f(\bar{a}, y)$ has at least λ real roots. The real roots are ordered by the increasing value and counted with multiplicities. A real algebraic number $\text{Root}_{y,\lambda}f \in \mathbb{R}$ given by a *defining polynomial* $f \in \mathbb{Z}[y]$ and a *root number* λ is the λ -th real root of f .

Let C be a connected subset of \mathbb{R}^k . The function $\text{Root}_{y,\lambda}f$ is *regular* on C if it is continuous on C , $c_d(\bar{a}) \neq 0$ for all $\bar{a} \in C$, and there exists $m \in \mathbb{N}_+$ such that for any $\bar{a} \in C$ $\text{Root}_{y,\lambda}f(\bar{a})$ is a root of $f(\bar{a}, y)$ of multiplicity m .

The polynomial f is *degree-invariant* on C if there exists $e \in \mathbb{N}$ such that $c_d(\bar{a}) = \dots = c_{e+1}(\bar{a}) = 0 \wedge c_e(\bar{a}) \neq 0$ for all $\bar{a} \in C$.

A set $W = \{f_1, \dots, f_m\}$ of polynomials is *delineable* on C if all elements of W are degree-invariant on C and for $1 \leq i \leq m$

$$f_i^{-1}(0) \cap (C \times \mathbb{R}) = \{r_{i,1}, \dots, r_{i,l_i}\}$$

where $r_{i,1}, \dots, r_{i,l_i}$ are disjoint regular semialgebraic functions and for $i_1 \neq i_2$ r_{i_1,j_1} and r_{i_2,j_2} are either disjoint or equal. Functions $r_{i,j}$ are *root functions of f_i over C* .

Let W be delineable on C , let $r_1 < \dots < r_l$ be all root functions of elements of W over C , and let $r_0 = -\infty$ and $r_{l+1} = \infty$. For $1 \leq i \leq l$, the i -th W -section over C is the set

$$\{(\bar{a}, a_{k+1}) : \bar{a} \in C \wedge a_{k+1} = r_i(\bar{a})\}$$

For $1 \leq i \leq l+1$, the i -th W -sector over C is the set

$$\{(\bar{a}, a_{k+1}) : \bar{a} \in C \wedge r_{i-1}(\bar{a}) < a_{k+1} < r_i(\bar{a})\}$$

W -stack over C is the set of all W -sections and W -sectors over C .

A formula F is an *algebraic constraint* with *bounds* $BDS(F)$ if it is a level- k equational or inequality constraint with $1 \leq k \leq n$ defined as follows.

- (1) A *level-1 equational constraint* has the form $x_1 = r$, where r is a real algebraic number, and $BDS(F) = \{r\}$.
- (2) A *level-1 inequality constraint* has the form $r_1 < x_1 < r_2$, where r_1 and r_2 are real algebraic numbers, $-\infty$, or ∞ , and $BDS(F) = \{r_1, r_2\} \setminus \{-\infty, \infty\}$.
- (3) A *level- $k+1$ equational constraint* has the form $x_{k+1} = r(\bar{x})$, where r is a semialgebraic function, and $BDS(F) = \{r\}$.
- (4) A *level- $k+1$ inequality constraint* has the form $r_1(\bar{x}) < x_{k+1} < r_2(\bar{x})$, where r_1 and r_2 are semialgebraic functions, $-\infty$, or ∞ , and $BDS(F) = \{r_1, r_2\} \setminus \{-\infty, \infty\}$.

A level- $k+1$ algebraic constraint F is *regular* on a connected set $C \subseteq \mathbb{R}^k$ if all elements of $BDS(F)$ are regular on C and, if F is an inequality constraint, $r_1 < r_2$ on C .

Definition 3. An atomic cylindrical algebraic formula (CAF) F in (x_1, \dots, x_n) has the form $F_1 \wedge \dots \wedge F_n$, where F_k is a level- k algebraic constraint for $1 \leq k \leq n$ and F_{k+1} is regular on the solution set of $F_1 \wedge \dots \wedge F_k$ for $1 \leq k < n$.

Level- k cylindrical formulas in (x_1, \dots, x_n) are defined recursively as follows

- (1) A *level- n cylindrical formula* is false or a disjunction of level- n algebraic constraints.
- (2) A *level- k cylindrical formula*, with $1 \leq k < n$, is false or has the form

$$(F_1 \wedge G_1) \vee \dots \vee (F_m \wedge G_m)$$

where F_i are level- k algebraic constraints and G_i are level- $k+1$ cylindrical formulas.

A cylindrical algebraic formula (CAF) is a level-1 cylindrical formula F such that distributing conjunction over disjunction in F gives

$$\text{DNF}(F) = F_1 \vee \dots \vee F_l$$

where each F_i is an atomic CAF. Let $C(F_i)$ denote the solution set of F_i and let $D(F) = \{C(F_1), \dots, C(F_l)\}$. The bound polynomials of F is a finite set $\text{BP}(F) \subset \mathbb{R}[x_1, \dots, x_n]$ which consists of all polynomials f such that $\text{Root}_{x_k, \lambda} f \in \text{BDS}(G)$ for some $1 \leq k \leq n$ and a level- k algebraic constraint G that appears in F .

Note that $C(F_i)$ is a cell and $D(F)$ is a finite collection of pairwise disjoint cylindrically arranged cells.

For a CAF F in (x_1, \dots, x_n) , let $\Pi_k(F)$ denote the CAF in (x_1, \dots, x_k) obtained from F by removing all level- $k+1$ subformulas. Then

$$D(\Pi_k(F)) = \{\Pi_k(C) : C \in D(F)\}$$

Following the terminology of [20], we define a well-based CAF as follows.

Definition 4. A CAF F is well-based if $D(F)$ is a $\text{BP}(F)$ -invariant CAD of \mathbb{R}^n and for any $f \in \text{BP}(F)$ if $f \in \mathbb{R}[x_1, \dots, x_{k+1}] \setminus \mathbb{R}[x_1, \dots, x_k]$ then for any $\bar{a} \in \mathbb{R}^k$ $f(\bar{a}, x_{k+1})$ is not identically zero.

In a CAD corresponding to a well-based CAF a closure of a cell is a union of cells and the only cells from other stacks adjacent to a given section are sections defined by the same polynomial. Moreover, any two adjacent cells have different dimensions and are connected through a chain of adjacent cells with dimensions increasing by one, and hence to determine all cell adjacencies it is sufficient to find all pairs of adjacent cells whose dimensions differ by one. These properties, stated precisely in Proposition 5, are essential for our algorithm.

Proposition 5. Let F be a well-based CAF.

- (1) If $C \in D(F)$ then there exists cells $C_1, \dots, C_m \in D(F)$ such that $\bar{C} = C \cup C_1 \cup \dots \cup C_m$.
- (2) Let $C \in D(F)$ be a section

$$C = \{(\bar{a}, a_n) : \bar{a} \in \Pi_{n-1}(C) \wedge a_n = \text{Root}_{x_n, \lambda} f(\bar{a})\}$$

and let $C' \in D(\Pi_{n-1}(F))$ be a cell adjacent to $\Pi_{n-1}(C)$ with $\dim C' < \dim \Pi_{n-1}(C)$. Then either $\bar{C} \cap (C' \times \mathbb{R})$ is equal to a section

$$\{(\bar{a}, a_n) : \bar{a} \in C' \wedge a_n = \text{Root}_{x_n, \lambda'} f(\bar{a})\}$$

for some $1 \leq \lambda' \leq \deg_{x_n}(f)$, or for any $\bar{a} \in C'$

$$\lim_{\bar{b} \in \Pi_{n-1}(C), \bar{b} \rightarrow \bar{a}} \text{Root}_{x_n, \lambda} f(\bar{a}) = -\infty$$

or for any $\bar{a} \in C'$

$$\lim_{\bar{b} \in \Pi_{n-1}(C), \bar{b} \rightarrow \bar{a}} \text{Root}_{x_n, \lambda} f(\bar{a}) = \infty$$

- (3) Let $C_k, C_l \in D(F)$ be adjacent cells such that $\dim(C_k) = k$ and $\dim(C_l) = l$. Then $k \neq l$ and if $k < l$ then there exist cells $C_{k+1}, \dots, C_{l-1} \in D(F)$ such that $\dim(C_j) = j$ and $C_j \subseteq \bar{C}_{j+1}$ for $k \leq j < l$.

Proof. Part (1) is Lemma 1 of [20]. To prove (2) first note that, by (1), $C' \subset \overline{\Pi_{n-1}(C)}$. By Lemma 2 of [20], there exists a unique continuous function $r : \overline{\Pi_{n-1}(C)} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ extending $\text{Root}_{x_n, \lambda} f(\bar{a})$. Moreover, r is either infinite or a root of f . Since f is delineable on C' and C' is connected, either r is a root of f on C' , or $r \equiv -\infty$ on C' , or $r \equiv \infty$ on C' .

We will prove (3) by induction on n . Note that by (1) it is sufficient to find cells C_j such that $\dim(C_j) = j$ and C_j and C_{j+1} are adjacent for $k \leq j < l$. If $n = 1$ then dimensions of any pair of adjacent cells differ by one, hence (3) is true. To prove (3) for $n > 1$ we will use induction on $l - k$. If $l - k = 1$ then (3) is true. Suppose $l - k > 1$. Let $C'_{k'} = \Pi_{n-1}(C_k)$ and $C'_{l'} = \Pi_{n-1}(C_l)$, where $\dim(C'_{k'}) = k'$ and $\dim(C'_{l'}) = l'$. If C_l is a section, then, by Lemma 2 of [20], there exists a continuous function $r : \overline{C'_{l'}} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ such that $C_l = \{(x, r(x)) : x \in C'_{l'}\}$ and r is infinite or a root of an element of $BP(F)$. In this case set $s = r$. Similarly, if C_l is a sector, then, by Lemma 2 of [20], there exists continuous functions $r, s : \overline{C'_{l'}} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ such that $C_l = \{(x, y) : x \in C'_{l'} \wedge r(x) < y < s(x)\}$ and r and s are infinite or roots of elements of $BP(F)$. Since $l - k > 1$, $k' < l'$. Suppose first that $l' - k' = 1$. Since $l - k > 1$, C_k is a section and C_l is a sector. If $C_k = \{(x, t(x)) : x \in C'_{k'}\}$, where $t = r$ or $t = s$, then t is finite on $C'_{k'}$, C_k is adjacent to $C_{k+1} := \{(x, t(x)) : x \in C'_{k'}\}$, and C_{k+1} is adjacent to C_l . Since $l = k + 2$, (3) is true. Otherwise, let C_{k+1} be the sector directly below C_k . Then $C_{k+1} \subset \{(x, y) : x \in C'_{k'} \wedge r(x) < y < s(x)\}$, and hence C_{k+1} is adjacent to C_l . Again, since $l = k + 2$, (3) is true. Now suppose that $l' - k' > 1$. $\Pi_{n-1}(F)$ is well-based, hence, by the inductive hypothesis on n , there exist cells $C'_{k'+1}, \dots, C'_{l'-1} \in D(\Pi_{n-1}(F))$ such that $\dim(C'_j) = j$ and $C'_j \subseteq \overline{C'_{j+1}}$ for $k' \leq j < l'$. Let $x_0 \in C'_{k'}$ and $(x_0, y_0) \in C_k$. Then $r(x_0) \leq y_0 \leq s(x_0)$. Since $C'_{k'}$ is adjacent to $C'_{l'-1}$, there exist a sequence $\{x_n\}_{n \geq 1}$ such that $x_n \in C'_{l'-1}$ and $\lim_{n \rightarrow \infty} x_n = x_0$. Put $y_n = \max(r(x_n), \min(s(x_n), y_0))$. Then $\lim_{n \rightarrow \infty} (x_n, y_n) = (x_0, y_0)$. The set

$$S = \{(x, y) : x \in C'_{l'-1} \wedge y \in \mathbb{R} \wedge r(x) \leq y \leq s(x)\}$$

is a union of a finite number of cells, $S \subset \overline{C_l}$, and $(x_n, y_n) \in S$. Hence, there exists a cell $C \subseteq S$ such that C contains infinitely many elements of the sequence $\{(x_n, y_n)\}_{n \geq 1}$. Therefore, C is adjacent to both C_k and C_l . Since $\dim C - k < l - k$ and $l - \dim C < l - k$, (3) is true by the inductive hypothesis on $l - k$. \square

For a given semialgebraic set A a well-based CAF F such that $D(F)$ is consistent with A may not exist in a given system of coordinates. However, as shown in [20], there always exists a linear change of variables after which a well-based CAF F such that $D(F)$ is consistent with A does exist.

Example 6. If A is the real solution set of $xy + xz + yz = 0$ then a well-based CAF F such that $D(F)$ is consistent with A does not exist for any order of variables. A CAD computed using McCallum's projection operator [14] includes cells

$$\begin{aligned} C_1 &= \{(x, y, z) : x > 0 \wedge y > -x \wedge z = -\frac{xy}{x+y}\} \\ C_2 &= \{(x, y, z) : x = 0 \wedge y = 0\} \end{aligned}$$

$\overline{C_1}$ is not a union of cells, since $\overline{C_1} \cap C_2 = \{(x, y, z) : x = 0 \wedge y = 0 \wedge z \geq 0\}$, and section C_1 is adjacent to a sector C_2 from a different stack. After the linear change of variables $(x, y, z) \rightarrow (x, y + z, z)$ A is transformed to the solution set of $z^2 + z(y + 2x) + xy = 0$. The following CAF F is well-based and $D(F)$ is consistent with the transformed A .

$$\begin{aligned} F &= (x < 0 \wedge G_1) \vee (x = 0 \wedge ((y < \text{Root}_{y,1g} \wedge G_1) \vee \\ &\quad (y = \text{Root}_{y,1g} \wedge G_2) \vee (y > \text{Root}_{y,1g} \wedge G_1))) \vee (x > 0 \wedge G_1) \end{aligned}$$

where

$$\begin{aligned}
f &= z^2 + z(y + 2x) + xy \\
g &= y^2 + 4x^2 \\
G_1 &= z < \text{Root}_{z,1}f \vee z = \text{Root}_{z,1}f \vee \text{Root}_{z,1}f < z < \text{Root}_{z,2}f \vee \\
&\quad z = \text{Root}_{z,2}f \vee z > \text{Root}_{z,2}f \\
G_2 &= z < \text{Root}_{z,1}f \vee z = \text{Root}_{z,1}f \vee z > \text{Root}_{z,1}f
\end{aligned}$$

3. ROOT ISOLATION ALGORITHMS

In this section we describe root isolation algorithms we will use in the algorithm computing cell adjacencies. Let us first introduce some notations and subalgorithms.

Let $\Delta(c, r) = \{z \in \mathbb{C} : |z - c| \leq r\}$ denote a disk in the complex plane, let $\mathbb{Q}_2 = \mathbb{Z}[\frac{1}{2}] = \{a2^b : a, b \in \mathbb{Z}\}$ denote the set of dyadic rational numbers, and let $I_2(\mathbb{C}) = \{\Delta(c, r) : c \in \mathbb{Q}_2[t] \wedge r \in \mathbb{Q}_2 \wedge r > 0\}$ denote the set of discs in the complex plane with dyadic Gaussian rational centers and dyadic rational radii. For a disc $Z = \Delta(c, r) \in I_2(\mathbb{C})$, let $\gamma(Z) := c$ and $\rho(Z) := r$ denote the center and the radius of Z , let $\underline{Z} := \max(0, |c| - r)$ and $\bar{Z} := |c| + r$ denote the minimum and maximum of absolute values of elements of Z , let $\text{conj}(Z)$ denote the disc that consists of complex conjugates of elements of Z , and let $\text{dbl}(Z) = \Delta(c, 2r)$ and $\text{quad}(Z) = \Delta(c, 4r)$. When we refer to interval arithmetic operations we mean circular complex interval (disc) arithmetic (see e.g. [18]).

Proposition 7. *There exists an algorithm (ApproximateRoots) that takes as input a polynomial*

$$g = b_N x^N + \dots + b_0 \in \mathbb{Q}_2[t][x]$$

and $p \in \mathbb{N}$ and outputs $(s_1, \dots, s_N) \in \mathbb{Q}_2[t]^N$ such that for any polynomial

$$f = a_N x^N + \dots + a_0 = a_N(x - \sigma_1) \cdots (x - \sigma_N) \in \mathbb{C}[x]$$

and any $\varepsilon > 0$ there exists $p_0 \in \mathbb{N}$ such that if $p \geq p_0$ and, for all $0 \leq i \leq N$,

$$|b_i - a_i| \leq 2^{-p} \max_{0 \leq i \leq N} |a_i|$$

then, after a suitable reordering of roots, for all $1 \leq j \leq N$ $|s_j - \sigma_j| \leq \varepsilon$.

Proof. The algorithm described in [17] satisfies Proposition 7. \square

Let us now describe a subalgorithm computing roots of polynomials with complex disc coefficients. The algorithm is based on the following proposition ([21], Proposition 4.1).

Proposition 8. *Let $f \in \mathbb{C}[z]$ be a polynomial of degree N , $z_0 \in \mathbb{C}$, $r > 0$, and let $c_i := |\frac{f^{(i)}(z_0)}{i!}|$. Suppose that*

$$\max_{0 \leq i < k} \left(\frac{Nc_i}{c_k} \right)^{\frac{1}{k-i}} < r < \min_{k < i \leq N} \left(\frac{c_k}{Nc_i} \right)^{\frac{1}{i-k}}$$

Then f has exactly k roots, multiplicities counted, in the disc $\Delta(z_0, r)$.

The following is an extended version of Algorithm 4.2 from [21]. The key difference is that this version does not assume that the leading coefficient does not contain zero and provides a lower bound on the absolute value of roots that tend to infinity when the leading coefficients that contain zero vanish.

Algorithm 9. (*IntervalRoots*)

Input: $Z_0, \dots, Z_N \in I_2(\mathbb{C})$.

Output: $D_1, \dots, D_m \in I_2(\mathbb{C})$, positive integers k_1, \dots, k_m , and a positive radius R , such that for any $a_0 \in Z_0, \dots, a_N \in Z_N$ and any $1 \leq i \leq m$ the polynomial $f = a_N z^N + \dots + a_0$ has exactly k_i roots in the disc D_i , multiplicities counted, and f has no roots in $\Delta(0, R) \setminus \bigcup_{i=1}^m D_i$. Moreover, for any $1 \leq i < j \leq m$, $D_i \cap D_j = \emptyset$, and $D_i \subseteq \Delta(0, R)$. The other possible output is *Failed*.

- (1) If $0 \in Z_i$ for all $0 \leq i \leq N$ return *Failed*. Otherwise let d be the maximal i such that $0 \notin Z_i$.

- (2) Put

$$R := \min_{d < i \leq N} \left(\frac{Z_d}{N Z_i} \right)^{\frac{1}{i-d}}$$

If

$$R \leq \max_{0 \leq i < d} \left(\frac{N \overline{Z_i}}{Z_d} \right)^{\frac{1}{d-i}}$$

return *Failed*.

- (3) Set $f_c = b_d z^d + \dots + b_0$, where $b_i = \gamma(Z_i)$ for $0 \leq i \leq d$, and set

$$p = - \max_{0 \leq i \leq d} \log \rho(Z_i)$$

- (4) Compute $(s_1, \dots, s_d) = \text{ApproximateRoots}(f_c, p)$.
- (5) Let $F = Z_N z^N + \dots + Z_0$. For each $1 \leq j \leq d$ and $0 \leq i \leq N$ use complex disc arithmetic to compute $W_{i,j} := \frac{F^{(i)}(s_j)}{i!}$.
- (6) For each $1 \leq j \leq d$ let \tilde{k}_j be the smallest $k > 0$ such that $W_{k,j} \neq 0$ and

$$r_j := \max_{0 \leq i < k} \left(\frac{N \overline{W_{i,j}}}{W_{k,j}} \right)^{\frac{1}{k-i}} < \min_{k < i \leq N} \left(\frac{W_{k,j}}{N \overline{W_{i,j}}} \right)^{\frac{1}{i-k}}$$

If there is no k satisfying the condition return *Failed*.

- (7) Find the connected components of the union of $\Delta(s_j, r_j)$. Let J_1, \dots, J_m be the sets of indices j corresponding to the connected components.
- (8) For each $1 \leq l \leq m$, let $J_l = \{j_{l,1}, \dots, j_{l,k_l}\}$. If for some $j \in J_l$ $\tilde{k}_j \neq k_l$ return *Failed*. Otherwise pick $j \in J_l$ with the minimal value of r_j , and set $D_l := \Delta(s_j, r_j)$.
- (9) If $D_l \not\subseteq \Delta(0, R)$ for some $1 \leq l \leq m$ return *Failed*.
- (10) Return (D_1, \dots, D_m) , (k_1, \dots, k_m) , and R .

To show correctness of Algorithm 9, suppose that the algorithm returned (D_1, \dots, D_m) , (k_1, \dots, k_m) , and R . Let $a_0 \in Z_0, \dots, a_N \in Z_N$ and $f = a_N z^N + \dots + a_0$. By Proposition 8, and because the algorithm did not fail in step (2), f has exactly d roots in $\Delta(0, R)$. The condition in step (6) and Proposition 8 imply that f has exactly k_i roots in the disc D_i . Since the algorithm did not fail in step (8), $k_1 + \dots + k_m = d$, and hence f has no roots in $\Delta(0, R) \setminus \bigcup_{i=1}^m D_i$. Step (7) guarantees that for any $1 \leq i < j \leq m$, $D_i \cap D_j = \emptyset$. Step (9) ensures that $D_i \subseteq \Delta(0, R/2)$.

Computation of sample points in CAD cells requires a representation of vectors with algebraic number coordinates. The following gives a recursive definition of root isolation data and of representation of real algebraic vectors. Note that root isolation data provides information about roots of f_k not only at u , but also in a neighbourhood of u (point (5) of the definition). This property is crucial for computing cell adjacencies.

Definition 10. $\Theta_k = ((D_1, \dots, D_m), (k_1, \dots, k_m), R)$ is root isolation data for $f_k \in \mathbb{Q}[x_1, \dots, x_k]$ at $u = (\alpha_1, \dots, \alpha_{k-1}) \in \mathbb{R}^{k-1}$ if

- (1) $D_1, \dots, D_m \in I_2(\mathbb{C})$, $k_1, \dots, k_m \in \mathbb{N}_+$, and $R > 0$,
- (2) $g_k = f_k(\alpha_1, \dots, \alpha_{k-1}, x_k)$ has a root of multiplicity k_j in D_j , for $1 \leq j \leq m$, and has no other roots,
- (3) for any $1 \leq j \leq m$ $\text{dbl}(D_j) \subseteq \Delta(0, R/2)$,
- (4) for any $j_1 \neq j_2$ $\text{dbl}(D_{j_1}) \cap \text{dbl}(D_{j_2}) = \emptyset$ and if $\text{dbl}(D_{j_1}) \cap \mathbb{R} \neq \emptyset$ then $\text{conj}(\text{dbl}(D_{j_1})) \cap \text{dbl}(D_{j_2}) = \emptyset$,
- (5) if W_i is the isolating disc of α_i for $1 \leq i < k$, and $\beta_i \in \text{quad}(W_i)$ then $f_k(\beta_1, \dots, \beta_{k-1}, x_k)$ has exactly k_j roots in D_j , multiplicities counted, and has no roots in $\Delta(0, R) \setminus \bigcup_{j=1}^m D_j$.

A real algebraic vector $v = \text{RAV}(\Omega_k) = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ is represented by

$$\Omega_k = (\Omega_{k-1}, f_k, W_k, \Theta_k)$$

where

- Ω_{k-1} represents the real algebraic vector $\Pi(v) = \text{RAV}(\Omega_{k-1}) = (\alpha_1, \dots, \alpha_{k-1}) \in \mathbb{R}^{k-1}$,
- $f_k \in \mathbb{Q}[x_1, \dots, x_k]$ is a defining polynomial of α_k ,
- $W_k \in I_2(\mathbb{C})$ is an isolating disc of α_k ,
- $\Theta_k = ((D_1, \dots, D_m), (k_1, \dots, k_m), R)$ is root isolation data for f_k at $\Pi(v)$, $W_k = D_{j_0}$ for some $1 \leq j_0 \leq m$, and $W_k \cap \mathbb{R} \neq \emptyset$.

Define $\rho(\Omega_k) = \max(\rho(\Omega_{k-1}), \max_{1 \leq j \leq m} \rho(D_j))$.

To complete the recursive definition let $\Omega_0 = ()$ be the representation of the only element of \mathbb{R}^0 .

We will say that $\Omega'_k = (\Omega'_{k-1}, f'_k, W'_k, \Theta'_k)$ is a refinement of Ω_k if $\text{RAV}(\Omega'_k) = \text{RAV}(\Omega_k)$, $\Theta'_k = ((D'_1, \dots, D'_m), (k_1, \dots, k_m), R')$, $\rho(D'_j) < \rho(D_j)$ and $D'_j \subseteq \text{dbl}(D_j)$ for $1 \leq j \leq m$, and Ω'_{k-1} is a refinement of Ω_{k-1} .

For any $v = (a_1, \dots, a_k) \in \mathbb{R}^k$ and $a \in \mathbb{R}$, we will use notation $\Lambda(v) = a_k$ and $v \times a = (a_1, \dots, a_k, a) \in \mathbb{R}^{k+1}$.

Remark 11. A refinement Ω''_k of a refinement Ω'_k of Ω_k is a refinement of Ω_k .

Proof. With notations from Definition 10, let

$$\Omega''_k = (\Omega''_{k-1}, f_k, W''_k, \Theta''_k)$$

and

$$\Theta''_k = ((D''_1, \dots, D''_m), (k_1, \dots, k_m), R'')$$

By induction, it suffices to show $\rho(D''_j) < \rho(D_j)$ and $D''_j \subseteq \text{dbl}(D_j)$ for $1 \leq j \leq m$. $\rho(D''_j) < \rho(D_j)$ follows from $\rho(D'_j) < \rho(D_j)$ and $\rho(D'_j) < \rho(D_j)$. $D'_j \subseteq \text{dbl}(D'_j)$ and $D'_j \subseteq \text{dbl}(D_j)$ implies that D_j , D'_j and D''_j contain the same root of $f_k(\alpha_1, \dots, \alpha_{k-1}, x_k)$. Then $\rho(D''_j) < \rho(D_j)$ implies $D''_j \subseteq \text{dbl}(D_j)$. \square

The algorithms we introduce next take a *working precision* argument. A working precision p is a positive integer. One can think of it as the number of bits in floating-point numbers used in a numeric approximation algorithm. However, we will not attach any specific meaning to the working precision argument. Instead our algorithms will satisfy certain properties as p tends to infinity. For instance, if we say that a certain quantity ω in the output of an algorithm tends to zero as p tends to infinity, it means that for any $\varepsilon > 0$

there exists $N > 0$ such that if the working precision $p > N$ then the algorithm will produce an output with $\omega < \varepsilon$.

Let $v = (\alpha_1, \dots, \alpha_k)$ be a real algebraic vector and let

$$f \in \mathbb{Q}[x_1, \dots, x_k, x_{k+1}]$$

be such that $f(\alpha_1, \dots, \alpha_k, x_{k+1})$ does not vanish identically. We will now describe an algorithm $AlgRoots_k$, with $k \geq 0$, which finds the root isolation data of f at v and the real roots of $f(\alpha_1, \dots, \alpha_k, x_{k+1})$. The algorithm uses two subalgorithms $Refine_k$ and $ZeroTest_k$ that will be defined recursively in terms of $AlgRoots_{k-1}$. Given

$$(\alpha_1, \dots, \alpha_k) = RAV(\Omega_k)$$

and a working precision $p > 0$ $Refine_k$ computes a refinement Ω'_k of Ω_k such that as p tends to infinity $\rho(\Omega'_k)$ tends to zero. $ZeroTest_k$ decides whether $h(\alpha_1, \dots, \alpha_k)$ is zero for a given $h \in \mathbb{Q}[x_1, \dots, x_k]$.

Algorithm 12. ($AlgRoots_k$)

Input: Real algebraic vector $v = (\alpha_1, \dots, \alpha_k) = RAV(\Omega_k)$, where $k \geq 0$, $f \in \mathbb{Q}[x_1, \dots, x_k, x_{k+1}]$, such that $f(\alpha_1, \dots, \alpha_k, x_{k+1})$ does not vanish identically, and a working precision $p > 0$.

Output: Root isolation data Θ of f at v , a refinement Ω'_k of Ω_k , and real algebraic vectors $v_1 = RAV(\Omega_{k+1,1}), \dots, v_r = RAV(\Omega_{k+1,r}) \in \mathbb{R}^{k+1}$ such that $\Pi(v_j) = RAV(\Omega'_k)$, for $1 \leq j \leq r$, $\Lambda(v_1), \dots, \Lambda(v_r)$ are all the real roots of $f(\alpha_1, \dots, \alpha_k, x_{k+1})$, and as p tends to infinity $\rho(\Omega_{k+1,j})$ tends to zero.

- (1) Let $f = a_N x_{k+1}^N + \dots + a_0$. Find d such that $a_d(\alpha_1, \dots, \alpha_k) \neq 0$ and

$$a_{d+1}(\alpha_1, \dots, \alpha_k) = \dots = a_N(\alpha_1, \dots, \alpha_k) = 0$$

using $ZeroTest_k$ if $k > 0$. Set $g = a_d z^d + \dots + a_0$.

- (2) Compute the principal subresultant coefficients $p_{sc0}, \dots, p_{sc_{n-1}}$ of g and $\frac{\partial g}{\partial z}$ with respect to z .

- (3) Find the largest integer $\mu \geq 0$ such that

$$p_{sc0}(\alpha_1, \dots, \alpha_k) = \dots = p_{sc_{\mu-1}}(\alpha_1, \dots, \alpha_k) = 0$$

using $ZeroTest_k$ if $k > 0$.

- (4) Set $p' = p$ and $\Omega'_k = \Omega_k$.
- (5) If $k > 0$ compute $\Omega'_k = Refine_k(\Omega'_k, p')$ and let W_1, \dots, W_k be the isolating discs of $\alpha_1, \dots, \alpha_k$ in Ω'_k .
- (6) For $0 \leq i \leq N$
 - (a) if $a_i \in \mathbb{Q}$ compute $Z_i \in I_2(\mathbb{C})$ such that $a_i \in Z_i$ and $\rho(a_i) \leq 2^{-p'}$,
 - (b) else compute $Z_i = a_i(\text{quad}(W_1), \dots, \text{quad}(W_k))$ using interval arithmetic.
- (7) Compute $\Theta := IntervalRoots(Z_0, \dots, Z_N)$.
- (8) If $\Theta = \text{Failed}$ double p' and go to step (5).
- (9) Let $\Theta = ((D_1, \dots, D_m), (k_1, \dots, k_m), R)$. If $d - m > \mu$ or the conditions (3) and (4) of Definition 10 are not satisfied, double p' and go to step (5).
- (10) Let (j_1, \dots, j_r) be the set of indices for which $D_{j_l} \cap \mathbb{R} \neq \emptyset$. For $1 \leq l \leq r$ let $\Omega_{k+1,l} = (\Omega'_k, f, D_{j_l}, \Theta)$ and $v_l = RAV(\Omega_{k+1,l})$.
- (11) Return Θ , Ω'_k , and v_1, \dots, v_r .

Proof. To prove termination of the Algorithm 12 we need to show that for sufficiently large p' the call to $IntervalRoots$ in step (7) succeeds and gives a result with $d - m = \mu$. The specification of the algorithm $Refine_k$ implies that as p' tends to infinity $\max_{1 \leq i \leq k} \rho(W_i)$ tends to zero. Hence also $\max_{1 \leq i \leq N} \rho(Z_i)$ tends to zero. Therefore, as p' tends to infinity,

$\gamma(Z_i)$ tends to $a_i(\alpha_1, \dots, \alpha_k)$ and Z_i and $\overline{Z_i}$ tend to $|a_i(\alpha_1, \dots, \alpha_k)|$ for all $0 \leq i \leq N$. In particular, for sufficiently large p' , $0 \notin Z_i$ iff $a_i(\alpha_1, \dots, \alpha_k) \neq 0$, and hence d in step (1) of *IntervalRoots* is the same as d computed in step (1) of Algorithm 12. Since

$$R := \min_{d < i \leq N} \left(\frac{Z_d}{N Z_i} \right)^{\frac{1}{i-d}}$$

tends to infinity and

$$\max_{0 \leq i < d} \left(\frac{N \overline{Z_i}}{Z_d} \right)^{\frac{1}{d-i}}$$

tends to a finite constant, the call to *IntervalRoots* does not fail in step (2) for sufficiently large p' . Let $\sigma_1, \dots, \sigma_d$ be the roots of $h(z) = g(\alpha_1, \dots, \alpha_k, z)$, each repeated as many times as its multiplicity. Let s_1, \dots, s_d be the roots computed in step (4) of *IntervalRoots*. The specification of *ApproximateRoots* implies that, as p' tends to infinity, after a suitable re-ordering of roots, s_j tends to σ_j for each $1 \leq j \leq d$. Hence for $W_{i,j} := \frac{F^{(i)}(s_j)}{i!}$ computed in step (5) of *IntervalRoots* $\gamma(W_{i,j})$ tends to $\frac{h^{(i)}(\sigma_j)}{i!}$ and $\rho(W_{i,j})$ tends to zero. Therefore, $\overline{W_{i,j}}$ and $\overline{W_{i,j}}$ tend to $|\frac{h^{(i)}(\sigma_j)}{i!}|$. Hence, if k_j is the multiplicity of σ_j ,

$$r_j := \max_{0 \leq i < k_j} \left(\frac{N \overline{W_{i,j}}}{W_{k_j,j}} \right)^{\frac{1}{k_j-i}}$$

tends to zero and

$$\min_{k_j < i \leq N} \left(\frac{W_{k_j,j}}{N \overline{W_{i,j}}} \right)^{\frac{1}{i-k_j}}$$

is bounded away from zero for sufficiently large p' . Therefore, for sufficiently large p' , the condition in step (6) of *IntervalRoots* is satisfied by k_j . Note that if s_j is closer to σ_j than to other roots of h then the condition cannot be satisfied by any $k < k_j$. Otherwise, by Proposition 8, h would have exactly k roots in $\Delta(s_j, r_j)$ which is impossible since if $\sigma_j \notin \Delta(s_j, r_j)$ then $\Delta(s_j, r_j)$ contains no roots of h and else $\Delta(s_j, r_j)$ contains at least k_j roots of h . Hence, for sufficiently large p' , step (6) does not fail and $\tilde{k}_j = k_j$ for each $1 \leq j \leq d$. Since r_j tends to zero for $1 \leq j \leq d$, for sufficiently large p' , $\Delta(s_{j_1}, r_{j_1})$ and $\Delta(s_{j_2}, r_{j_2})$ intersect iff $\sigma_{j_1} = \sigma_{j_2}$, and hence step (8) of *IntervalRoots* does not fail. Since R tends to infinity, for sufficiently large p' , step (9) of *IntervalRoots* does not fail and the whole algorithm succeeds. Since, for sufficiently large p' , the discs returned by *IntervalRoots* correspond to distinct roots of h , $d - m = \mu$ in step (9) of Algorithm 12. Since $\rho(D_i)$ tends to zero for $1 \leq i \leq m$ and R tends to infinity, for sufficiently large p' , we have $dbl(D_i) \subseteq \Delta(0, R/2)$ for any $1 \leq i \leq m$ and

$$\rho(D_i) < \frac{\min_{\sigma_{j_1} \neq \sigma_{j_2}} |\sigma_{j_1} - \sigma_{j_2}|}{16}$$

Therefore for any $i_1 \neq i_2$ $dbl(D_{i_1}) \cap dbl(D_{i_2}) = \emptyset$ and if σ is the root of h in D_{i_1} then either $\sigma \notin \mathbb{R}$ and $dbl(D_{i_1}) \cap \mathbb{R} = \emptyset$ or $\sigma \in \mathbb{R}$ and $conj(dbl(D_{i_1})) \cap dbl(D_{i_2}) = \emptyset$. Hence for sufficiently large p' , the conditions (3) and (4) of Definition 10 are satisfied and the algorithm terminates.

Proposition 4.3 of [21] and correctness of Algorithm 9 imply that $\Omega_{k+1,l}$ satisfy the conditions (1), (2), and (5) of Definition 10, and the conditions (3) and (4) are ensured by step (9) of Algorithm 12. Step (10) selects all isolating discs that intersect the real line, hence $\Lambda(v_1), \dots, \Lambda(v_r)$ are all the real roots of $f(\alpha_1, \dots, \alpha_k, x_{k+1})$.

To show that $\rho(\Omega_{k+1,j})$ tends to zero as p tends to infinity, note that $p' \geq p$, we have already shown that $\rho(D_i)$ tends to zero for $1 \leq i \leq m$ as p' tends to infinity, and since Ω'_k is computed by $Refine_k$ with working precision p' , $\rho(\Omega'_k)$ tends to zero as p' tends to infinity. \square

Remark 13. If $m = d$ in step (10) of Algorithm 12 then

$$g(\alpha_1, \dots, \alpha_k, z)$$

does not have multiple roots and computing the principal subresultant coefficients is not necessary. Hence instead of computing the principal subresultant coefficients in step (2) it is sufficient to compute them only when the algorithm reaches step (9) for the first time and $m < d$.

To complete the description of Algorithm 12 let us now define the subalgorithms $Refine_k$ and $ZeroTest_k$.

Algorithm 14. ($Refine_k$)

Input: Real algebraic vector $(\alpha_1, \dots, \alpha_k) = RAV(\Omega_k)$, where $k \geq 1$, and a working precision $p > 0$.

Output: A refinement Ω'_k of Ω_k such that as p tends to infinity $\rho(\Omega'_k)$ tends to zero.

- (1) Let $\Omega_k = (\Omega_{k-1}, f_k, W_k, \Theta_k)$,

$$\Theta_k = ((D_1, \dots, D_m), (k_1, \dots, k_m), R)$$

and $W_k = D_{j_0}$. Set $v = RAV(\Omega_{k-1})$ and $p' = p$.

- (2) Compute

$$(\Theta; \Omega'_{k-1}; v_1, \dots, v_r) = AlgRoots_{k-1}(v, f_k, p')$$

where $\Theta = ((D'_1, \dots, D'_m), (k_1, \dots, k_m), R')$.

- (3) If no reordering of indices in Θ yields $\rho(D'_j) < \rho(D_j)$ and $D'_j \subseteq dbl(D_j)$ for $1 \leq j \leq m$, double p' and go to (2).

- (4) Let $v_j = RAV(\Omega'_k)$ be such that the isolating disk of $\Lambda(v_j)$ is D'_{j_0} . Return Ω'_k .

Since $\rho(D'_j)$ tends to zero as p' tends to infinity, for sufficiently large p' the condition in step (3) is satisfied for pairs D_j and D'_j containing the same root of $f_k(\alpha_1, \dots, \alpha_{k-1}, x_k)$, and hence the algorithm terminates. Correctness of Algorithm 12 and the condition in step (3) guarantee that Ω'_k is a refinement of Ω_k and as p tends to infinity $\rho(\Omega'_k)$ tends to zero.

Algorithm 15. ($ZeroTest_k$)

Input: Real algebraic vector $(\alpha_1, \dots, \alpha_k) = RAV(\Omega_k)$, where $k \geq 1$, and $h \in \mathbb{Q}[x_1, \dots, x_k]$.

Output: true if $h(\alpha_1, \dots, \alpha_k) = 0$ and false otherwise.

- (1) Let $\Omega_k = (\Omega_{k-1}, f_k, W_k, \Theta_k)$,

$$\Theta_k = ((D_1, \dots, D_m), (k_1, \dots, k_m), R)$$

and $W_k = D_{j_0}$. Set $\mu = k_{j_0}$, $\Omega'_k = \Omega_k$, and set an initial value p of working precision (e.g. to precision that was used to compute Ω_k).

- (2) Compute $\Omega'_k = Refine_k(\Omega'_k, p)$. Let $\Omega'_k = (\Omega'_{k-1}, f_k, W'_k, \Theta'_k)$. Set $v = RAV(\Omega'_{k-1})$.

- (3) Compute $(\Theta; \Omega'_{k-1}; v_1, \dots, v_r) = AlgRoots_{k-1}(v, f_k h, p)$.

- (4) If W'_k intersects the isolating disc of $\Lambda(v_j)$ for more than one j , double p and go to step (2).

- (5) Let j be the only index for which W'_k intersects the isolating disc W of $\Lambda(v_j)$. Let $\Theta = ((\tilde{D}_1, \dots, \tilde{D}_m), (\tilde{k}_1, \dots, \tilde{k}_m), \tilde{R})$, and $W = \tilde{D}_{j_0}$.

- (6) If $k_{j_0} > \mu$ return true otherwise return false.

Since as p' tends to infinity, $\rho(W'_k)$ and $\max_{1 \leq j \leq r} \rho(\Lambda(v_j))$ tend to zero, for sufficiently large p' , W'_k intersects only the isolating disc of $\Lambda(v_j) = \alpha_k$, which proves termination and correctness of the algorithm.

Since $AlgRoots_k$ is defined for $k \geq 0$ and $Refine_k$ and $ZeroTest_k$ are defined for $k \geq 1$, the recursive definition of the algorithms is complete.

Remark 16. *ZeroTest_k is defined here in terms of AlgRoots_{k-1} for simplicity of description. In practice, to decide whether*

$$h(\alpha_1, \dots, \alpha_k) = 0$$

we can first evaluate h at the isolating discs of $\alpha_1, \dots, \alpha_k$ using interval arithmetic. If the result does not contain zero then

$$h(\alpha_1, \dots, \alpha_k) \neq 0$$

Otherwise, we isolate roots of

$$h_\alpha = h(\alpha_1, \dots, \alpha_{k-1}, z)$$

and refine isolating discs of roots of

$$g_\alpha = g(\alpha_1, \dots, \alpha_{k-1}, z)$$

and roots of h_α until either the isolating disc of α_k does not intersect any isolating discs of roots of h_α or the number of intersecting isolating discs of roots of g_α and h_α agrees with the number of common roots of g_α and h_α computed by finding signs of principal subresultant coefficients of g_α and h_α (see Proposition 4.4 of [21]). When the algorithm is used in CAD construction we also use information about polynomials that are zero at the current point that was collected during the construction (see [21], Section 4.1).

4. FINDING CELL ADJACENCIES

Let F be a well-based CAF in x_1, \dots, x_n . For simplicity let us assume that $BP(F) = \{f_1, \dots, f_n\}$, where

$$f_k \in \mathbb{Q}[x_1, \dots, x_k] \setminus \mathbb{Q}[x_1, \dots, x_{k-1}]$$

This can be always achieved by multiplying all elements of

$$BP(F) \cap (\mathbb{Q}[x_1, \dots, x_k] \setminus \mathbb{Q}[x_1, \dots, x_{k-1}])$$

We can also assume that f_1, \dots, f_n are square-free.

The main algorithm *CADAdjacency* (Algorithm 18) finds all pairs of adjacent cells $(C, C') \in D(F)^2$ such that $\dim C - \dim C' = 1$. By Proposition 5, to determine all cell adjacencies for a well-based CAF it is sufficient to find all pairs of adjacent cells whose dimensions differ by one, hence Algorithm 18 is sufficient to fully solve the cell adjacency problem for well-based CAF.

The algorithm first calls *SamplePoints* (Algorithm 19), which constructs a sample point $SPT(C) = (a_1, \dots, a_k) \in \mathbb{R}^k$ in each cell $C \in D(\Pi_k(F))$, for $1 \leq k \leq n$, and computes root isolation data $RTS(C)$ for each cell $C \in D(\Pi_k(F))$, for $1 \leq k < n$. Let us describe the representation of sample points and give the specification of root isolation data. Let $I = (i_1, \dots, i_l)$ be the set of indices $1 \leq i \leq k$ such that $\Pi_i(C)$ is a section. For $i \notin I$, a_i is a rational number and for $i \in I$, a_i is an algebraic number. To represent sample points we will use combinations of rational vectors and algebraic vectors defined as follows. Let $1 \leq k \leq n$, let $0 \leq l \leq k$, let $I = \{i_1, \dots, i_l\}$, where $1 \leq i_1 < \dots < i_l \leq k$, and let $J = \{1, \dots, k\} \setminus I = \{j_1, \dots, j_{k-l}\}$, where $1 \leq j_1 < \dots < j_{k-l} \leq k$. Let $v = (\alpha_1, \dots, \alpha_l) = RAV(\Omega_l)$ be a real algebraic vector and let $w = (q_1, \dots, q_{k-l}) \in \mathbb{Q}^{k-l}$ be a rational vector. By $PT(v, w, I, J)$ we

denote the point $(a_1, \dots, a_k) \in \mathbb{R}^k$ such that $a_{i_s} = \alpha_s$ for $1 \leq s \leq l$ and $a_{j_t} = q_t$ for $1 \leq t \leq k-l$. Suppose $1 \leq k < n$ and $SPT(C) = PT(v, w, I, J)$. Let $f_{k+1}^C \in \mathbb{Q}[x_{i_1}, \dots, x_{i_l}, x_{k+1}]$ denote f_{k+1} with x_{j_t} replaced by a_{j_t} for $1 \leq t \leq k-l$. Then $RTS(C)$ computed by *SamplePoints* is root isolation data of f_{k+1}^C at v .

Next *CADAdjacency* calls *AdjacencyPoints* (Algorithm 20) which, for $1 \leq k \leq n$, and for each pair of adjacent cells (C, C') of $D(\Pi_k(F))$ with $\dim C' = \dim C - 1$, constructs a point $ADP(C, C') \in C$ which satisfies the following condition.

Condition 17. *If $SPT(C') = (a_1, \dots, a_k)$ and $ADP(C, C') = (b_1, \dots, b_k)$ then*

- *for each $1 \leq i \leq k$ if a_i is a root of $f_i(a_1, \dots, a_{i-1}, x_i)$ with isolating disc W_i then $b_i \in \text{dbl}(W_i)$,*
- *if a_i is a rational number between roots of $f_i(a_1, \dots, a_{i-1}, x_i)$ then $b_i = a_i$.*

Finally, *CADAdjacency* returns the pairs of cells $(C, C') \in D(F)^2$ for which $ADP(C, C')$ is defined.

Algorithm 18. (*CADAdjacency*)

Input: A well-based CAF F in x_1, \dots, x_n with $BP(F) = \{f_1, \dots, f_n\}$, where $f_k \in \mathbb{Q}[x_1, \dots, x_k] \setminus \mathbb{Q}[x_1, \dots, x_{k-1}]$.

Output: The set A of all pairs of adjacent cells $(C, C') \in D(F)^2$ such that $\dim C - \dim C' = 1$.

- (1) Compute $(SPT, RTS) = \text{SamplePoints}(F)$.
- (2) Compute $ADP = \text{AdjacencyPoints}(F, SPT, RTS)$.
- (3) Return the set of all pairs of cells $(C, C') \in D(F)^2$ such that $ADP(C, C')$ is defined.

Algorithm 19. (*SamplePoints*)

Input: A well-based CAF F in x_1, \dots, x_n with $BP(F) = \{f_1, \dots, f_n\}$, where $f_k \in \mathbb{Q}[x_1, \dots, x_k] \setminus \mathbb{Q}[x_1, \dots, x_{k-1}]$.

Output: SPT and RTS such that

- *for $1 \leq k \leq n$ and for each cell C of $D(\Pi_k(F))$, $SPT(C)$ is a sample point in C ,*
- *for $1 \leq k < n$ and for each cell C of $D(\Pi_k(F))$, $RTS(C)$ is root isolation data for C .*

- (1) Set an initial value p of working precision.
- (2) Compute $(\Theta; (); v_1, \dots, v_r) = \text{AlgRoots}_0((), f_1, p)$. We have

$$\Theta = ((D_1, \dots, D_m), (k_1, \dots, k_m), R)$$

- (3) Pick rational numbers $-R < q_1 < \Lambda(v_1) < q_2 < \dots < q_r < \Lambda(v_r) < q_{r+1} < R$ such that $q_i \notin \bigcup_{j=1}^m \text{dbl}(D_j)$.
- (4) For $1 \leq i \leq r$, let C be the i -th $\{f_1\}$ -section. Set $SPT(C) = PT(v_i, (), \{1\}, \{\})$.
- (5) For $1 \leq i \leq r+1$, let C be the i -th $\{f_1\}$ -sector. Set $SPT(C) = PT((), (q_i), \{\}, \{1\})$.
- (6) For $1 \leq k < n$ and for each cell C of $D(\Pi_k(F))$:
 - (a) Let $(a_1, \dots, a_k) = PT(v, w, I, J) = SPT(C)$.
 - (b) Let $I = \{i_1, \dots, i_l\}$, $J = \{j_1, \dots, j_{k-l}\}$, and let f_{k+1}^C be f_{k+1} with x_{j_t} replaced by a_{j_t} for $1 \leq t \leq k-l$.
 - (c) Compute $(\Theta; \Omega'_l; v_1, \dots, v_r) = \text{AlgRoots}_l(v, f_{k+1}^C, p)$. We have

$$\Theta = ((D_1, \dots, D_m), (k_1, \dots, k_m), R)$$

- (d) Set $RTS(C) = \Theta$ and replace representations of algebraic vectors in $SPT(\Pi_i(C))$ for $i \leq k$ with their refinements that appear in Ω'_l .

- (e) Pick rational numbers $-R < q_1 < \Lambda(v_1) < q_2 < \dots < q_r < \Lambda(v_r) < q_{r+1} < R$ such that $q_i \notin \bigcup_{j=1}^m \text{dbl}(D_j)$, and let $w_i = w \times q_i$ for $1 \leq i \leq r+1$.

- (f) For $1 \leq i \leq r$, let S be the i -th $\{f_{k+1}\}$ -section over C . Set

$$SPT(S) = PT(v_i, w, I \cup \{k+1\}, J)$$

- (g) For $1 \leq i \leq r+1$, let S be the i -th $\{f_{k+1}\}$ -sector over C . Set

$$SPT(S) = PT(v, w_i, I, J \cup \{k+1\})$$

- (7) Return SPT and RTS .

Algorithm 20. (*AdjacencyPoints*)

Input: A well-based CAF F in x_1, \dots, x_n with $BP(F) = \{f_1, \dots, f_n\}$, where $f_k \in \mathbb{Q}[x_1, \dots, x_k] \setminus \mathbb{Q}[x_1, \dots, x_{k-1}]$, SPT and RTS as in the output of Algorithm 19.

Output: ADP such that for $1 \leq k \leq n$ and for each pair of adjacent cells (C, C') of $D(\Pi_k(F))$ with $\dim C' = \dim C - 1$, $ADP(C, C')$ is a point in C satisfying Condition 17.

- (1) Let r be the number of real roots of f_1 . For $1 \leq i \leq r+1$:

- (a) Let C be the i -th $\{f_1\}$ -sector.

- (b) If $i > 1$ let C' be the $i-1$ -st $\{f_1\}$ -section. We have $SPT(C') = PT(v, (), \{1\}, \{\})$, $v = (\alpha) = \text{RAV}(\Omega_1)$, and $\Omega_1 = ((), f_1, W, \Theta)$. Let $q \in \text{dbl}(W) \cap \mathbb{Q}$ and $q > \alpha$. Set $ADP(C, C') = (q)$.

- (c) If $i \leq r$ let C' be the i -th $\{f_1\}$ -section. We have

$$SPT(C') = PT(v, (), \{1\}, \{\})$$

$$v = (\alpha) = \text{RAV}(\Omega_1), \text{ and}$$

$$\Omega_1 = ((), f_1, W, \Theta)$$

Let $q \in \text{dbl}(W) \cap \mathbb{Q}$ and $q < \alpha$. Set $ADP(C, C') = (q)$.

- (2) For $1 \leq k < n$ and for each non-zero-dimensional cell C of $D(\Pi_k(F))$:

- (a) Let $SPT(C) = PT(v, w, I, J)$ and let r be the number of real roots of f_{k+1} over C . For $1 \leq i \leq r+1$:

- (i) Let S be the i -th $\{f_{k+1}\}$ -sector over C .

- (ii) If $i > 1$ let S' be the $i-1$ -st $\{f_{k+1}\}$ -section over C . We have $SPT(S') = PT(v', w, I \cup \{k+1\}, J)$, $v' = v \times \alpha = \text{RAV}(\Omega_{l+1})$, and $\Omega_{l+1} = (v, g, W, \Theta)$. Let $q \in \text{dbl}(W) \cap \mathbb{Q}$ and $q > \alpha$. Set $ADP(S, S') = PT(v, w', I, J \cup \{k+1\})$, where $w' = w \times q$.

- (iii) If $i \leq r$ let S' be the i -th $\{f_{k+1}\}$ -section over C . We have $SPT(S') = PT(v', w, I \cup \{k+1\}, J)$, $v' = v \times \alpha = \text{RAV}(\Omega_{l+1})$, and $\Omega_{l+1} = (v, g, W, \Theta)$. Let $q \in \text{dbl}(W) \cap \mathbb{Q}$ and $q < \alpha$. Set

$$ADP(S, S') = PT(v, w', I, J \cup \{k+1\})$$

where $w' = w \times q$.

- (b) For each cell C' of $D(\Pi_k(F))$ adjacent to C and such that $\dim C' = \dim C - 1$:

- (i) Let $(a_1, \dots, a_k) = PT(v, w, I, J) = ADP(C, C')$ and let

$$RTS(C') = ((D_1, \dots, D_m), (k_1, \dots, k_m), R)$$

- (ii) Let S'_1, \dots, S'_s be the $\{f_{k+1}\}$ -sections over C' , and let W'_j be the isolating disc of $\Lambda(SPT(S'_j))$ for $1 \leq j \leq s$.

- (iii) Let $I = \{i_1, \dots, i_l\}$, $J = \{j_1, \dots, j_{k-l}\}$, and let $g \in \mathbb{Q}[x_{i_1}, \dots, x_{i_l}, x_{k+1}]$ be f_{k+1} with x_{j_t} replaced by a_{j_t} for $1 \leq t \leq k-l$.

- (iv) Compute $(\Theta; \Omega'_j; v_1, \dots, v_r) = \text{AlgRoots}_l(v, g, p)$.

- (v) For $1 \leq i \leq r$ refine the isolating disc W_i of $\Lambda(v_i)$ until it is contained in one of $dbl(W'_1), \dots, dbl(W'_s)$ or $W_i \cap \Delta(0, R/2) = \emptyset$. Let S be the i -th $\{f_{k+1}\}$ -section over C . If $W_i \subseteq dbl(W'_j)$, set $ADP(S, S'_j) = PT(v_i, w, I \cup \{k+1\}, J)$, and set $L(i) = S'_j$. Otherwise if $\Lambda(v_i) < 0$ set $L(i) = -\infty$ else set $L(i) = \infty$.
 - (vi) Set $L(0) = -\infty$ and $L(r+1) = \infty$.
 - (vii) For $1 \leq i \leq r+1$, let S be the i -th $\{f_{k+1}\}$ -sector over C . For each $\{f_{k+1}\}$ -sector S' over C' that lies between $L(i-1)$ and $L(i)$ put $u = w \times \Lambda(SPT(S'))$ and set $ADP(S, S') = PT(v, u, I, J \cup \{k+1\})$.
- (3) Return ADP .

Proof. Let us now prove correctness of Algorithm 18. The working precision p set in step (1) of *SamplePoints* is used in calls to *AlgRoots*. Since *AlgRoots* raises precision as needed to reach its goals, p is just an initial value and can be set arbitrarily e.g. to the number of bits in a double precision number. Steps (2)-(6) construct sample points $SPT(C)$ is all cells of $D(F)$, starting with sample points in cells of $D(\Pi_1(F))$, and then extending them to sample points in $D(\Pi_k(F))$ one coordinate at a time. An important fact to note is that isolating discs in the representations of already constructed sample points may change during the execution of step (6). Namely, in step (6d) the isolating discs of the coordinates of the sample points $SPT(\Pi_i(C))$ for all projections of the cell C are replaced with their refinements that were computed in the process of isolating the roots of f_{k+1}^C . In particular, for any cell $C \in D(F)$ if $SPT(C) = (a_1, \dots, a_n)$ then for any $1 \leq k \leq n$ $SPT(\Pi_k(C)) = (a_1, \dots, a_k)$ and the isolating discs that appear in the representations of any algebraic coordinate a_i in $SPT(C)$ and in $SPT(\Pi_k(C))$ are equal. Note however, that after *SamplePoints* is finished the representations of $SPT(C)$ are fixed.

In step (1) of *AdjacencyPoints* for each pair of adjacent cells (C, C') of $D(\Pi_1(F))$ with $\dim C' = \dim C - 1$ the algorithm constructs a point

$$ADP(C, C') = (q) \in C$$

such that if $SPT(C') = (\alpha)$ and W is the isolating disc of α then $q \in dbl(W)$. At the start of each iteration of the loop in step (2) the algorithm has already constructed a point $ADP(C, C')$ for each pair of adjacent cells (C, C') of $D(\Pi_k(F))$ with $\dim C' = \dim C - 1$. The points satisfy Condition 17. Steps (2a) and (2b) construct points $ADP(S, S')$ for each pair of adjacent cells (S, S') of $D(\Pi_{k+1}(F))$ with $\dim S' = \dim S - 1$. It is clear that the constructed points satisfy Condition 17. What we need to show is that the construction will always succeed, pairs of cells (S, S') for which $ADP(S, S')$ is constructed are adjacent, and $ADP(S, S')$ is constructed for every pair of adjacent cells (S, S') of $D(\Pi_{k+1}(F))$ with $\dim S' = \dim S - 1$. Step (2a) constructs $ADP(S, S')$ for every pair of adjacent cells from a stack over the same cell C . Note that in step (2a) we have $\alpha \in W$ and $(dbl(W) \setminus W) \cap \mathbb{R}$ consists of two intervals, one on each side of α , hence we can pick rational numbers $q \in dbl(W)$ with $q > \alpha$ or $q < \alpha$. If cells S and S' from stacks over different cells C and C' are adjacent and $\dim S' = \dim S - 1$ then C and C' must be adjacent and, by Proposition 5, $\dim C' = \dim C - 1$ and S is a section iff S' is a section. This shows that step (2) constructs $ADP(S, S')$ for every pair of adjacent cells (S, S') of $D(\Pi_{k+1}(F))$ with $\dim S' = \dim S - 1$.

Let us prove that the construction in step (2b) will always succeed and pairs of cells (S, S') for which $ADP(S, S')$ is constructed are adjacent. With notation of step (2b), let

$$(a_1, \dots, a_k) = PT(v, w, I, J) = ADP(C, C')$$

and

$$(a'_1, \dots, a'_k) = PT(v', w', I', J') = SPT(C')$$

Note that $J' \subseteq J$ and $a'_j = a_j$ for $j \in J'$. Let $g' = f_{k+1}^{C'}$ be f_{k+1} with x_j replaced by $a'_j = a_j$ for $j \in J'$. Then g is equal to g' with x_j replaced by a_j for $j \in J \cap I'$. For $i \in I'$ let U'_i be the isolating disk of a'_i in v' and let U_i be the isolating disk of a'_i in the representation of $\Pi(v')$ with which $RTS(C')$ was computed. Note that, by Remark 11, the current representation of v' is a refinement of the representation with which $RTS(C')$ was computed, hence $U'_i \subseteq dbl(U_i)$. Hence, $a_i \in dbl(U'_i)$ and $a_i \in quad(U_i)$. By the condition (5) of Definition 10, for each $1 \leq i \leq r$ either $\Lambda(v_i)$ belongs to one of W'_1, \dots, W'_s or $\Lambda(v_i) \notin \Delta(0, R)$. Therefore we can refine the isolating disc W_i of $\Lambda(v_i)$ so that it is contained in one of $dbl(W'_1), \dots, dbl(W'_s)$ or $W_i \cap \Delta(0, R/2) = \emptyset$. In the former case the i -th $\{f_{k+1}\}$ -section over C is adjacent to the j -th $\{f_{k+1}\}$ -section over C' , in the latter case the i -th $\{f_{k+1}\}$ -section over C tends to infinity whose sign is determined by the sign of $\Lambda(v_i)$. This shows that sections (S, S') for which $ADP(S, S')$ is constructed are adjacent. Finally, let S and S' be sectors over C and C' defined in step (2b(vii)). Then, by construction in step (6e) of *SamplePoints*, $q = \Lambda(SPT(S')) \notin \bigcup_{j=1}^m dbl(D_j)$, and since $W'_j \subseteq dbl(D_j)$ (possibly after reordering of indices), $q \notin \bigcup_{j=1}^s W'_j$. Moreover, $-R < q < R$. Since for each $1 \leq i \leq r$ either $\Lambda(v_i)$ belongs to one of W'_1, \dots, W'_s or $\Lambda(v_i) \notin \Delta(0, R)$, the point $PT(v, u, I, J \cup \{k+1\})$ defined in step (2b(vii)) belongs to S . If $ADP(S, S')$ is constructed in step (2b(vii)) then S' is a sector that lies between sections adjacent to the sections bounding S , hence S and S' are adjacent. \square

5. EMPIRICAL RESULTS

An algorithm computing CAD cell adjacencies has been implemented in C, as a part of the kernel of *Mathematica*. The implementation takes a quantified system of polynomial equations and inequalities S and uses *Mathematica* multi-algorithm implementation of CAD to compute a CAF F such that $D(F)$ is a CAD of \mathbb{R}^n consistent with the solution set A of S . If F is well-based the implementation uses Algorithm 20 to find the cell adjacencies. The implementation is geared towards solving a specified topological problem, e.g. finding the boundary or the connected components of A , hence it avoids computing cell adjacencies for cells that are known not to belong to the closure of A . The current implementation also works for non-well-based problems in \mathbb{R}^3 using ideas from [1] to extend Algorithm 20.

The experiments have been conducted on a Linux laptop with a 4-core 2.7 GHz Intel Core i7 processor and 16 GB of RAM. The reported CPU time is a total from all cores used. For each example we give three timings. t_{CAD} is the computation time of constructing a CAF consistent with the solution set the input system. t_{SP} is the time of refining the CAD to a $BP(F)$ -invariant CAD of \mathbb{R}^n and of constructing sample points in the CAD cells (steps (1)-(6) of Algorithm 20). Our implementation refines the CAD while constructing sample points, which is why we cannot give separate timings. The third timing, t_{ADJ} is the time of computing cell adjacency information (steps (7)-(9) of Algorithm 20). We also report the dimension \dim of the embedding space, the number N_{CELL} of cells in the CAD of A , the number N_{ADJ} of computed pairs of adjacent cells whose dimensions differ by one, and the number N_{CC} of connected components of A .

Example 21. Find cell adjacencies for a CAD of the union of two unit balls in \mathbb{R}^n

$$x_1^2 + \dots + x_n^2 \leq 1 \vee (x_1 - 1)^2 + \dots + (x_n - 1)^2 \leq 1$$

TABLE 1. Union of two balls in \mathbb{R}^n

| dim | t_{CAD} | t_{SP} | t_{ADJ} | N_{CELL} | N_{ADJ} | N_{CC} |
|-----|-----------|----------|-----------|------------|-----------|----------|
| 2 | 0.018 | 0.004 | 0.001 | 21 | 42 | 1 |
| 3 | 0.100 | 0.033 | 0.024 | 179 | 718 | 1 |
| 4 | 0.489 | 0.175 | 0.112 | 521 | 3898 | 1 |
| 5 | 1.42 | 0.773 | 0.352 | 954 | 11910 | 2 |
| 6 | 44.6 | 24.8 | 8.92 | 14050 | 251758 | 2 |

TABLE 2. Wilson's benchmark

| Ex # | dim | t_{CAD} | t_{SP} | t_{ADJ} | N_{CELL} | N_{ADJ} | N_{CC} |
|------|-----|-----------|----------|-----------|------------|-----------|----------|
| 2.13 | 4 | 0.109 | 0.263 | 0.210 | 3104 | 10576 | 1 |
| 2.16 | 3 | 3.06 | 2.65 | 1.27 | 2811 | 37416 | 1 |
| 6.1 | 3 | 0.768 | 0.794 | 0.312 | 2774 | 8926 | 2 |
| 5.10 | 4 | 14.9 | 11.2 | 4.01 | 2256 | 63190 | 1 |
| 6.6 | 6 | 14.6 | 7.85 | 3.52 | 2128 | 76360 | 1 |

Note that for $n \leq 3$ the balls have full-dimensional intersection, for $n = 4$ they touch at one point, and for $n > 4$ they are disjoint.

Example 22. Here we used modified versions of examples from Wilson's benchmark set [26] (version 4). Of the 77 examples we selected 63 that involved at least 3 variables and we used quantifier-free versions of the examples. In 21 of the examples the system was not well-based and involved more than 3 variables, hence our algorithm did not apply. 7 examples did not finish in 600 seconds. Of the 35 examples for which our implementation succeeded, 29 were well-based and 6 were not well-based and in \mathbb{R}^3 . On average, t_{CAD} took 55% of the total time, t_{SP} took 34%, and t_{ADJ} took 11%. Five examples with the largest number of cells are given in Table 2. All but the third example are well-based.

Example 23. Here we took the 32 3D solids that appear in Mathematica SOLIDDATA and intersected each of them with the solution set of $9(x + y + z)^2 > z^2 + 1$. All 32 examples were well-based and in all our implementation succeeded. On average, t_{CAD} took 37% of the total time, t_{SP} took 47%, and t_{ADJ} took 16%. The five solids which resulted in the largest number of cells are:

(1) Steinmetz 6-solid

$$\begin{aligned} 2x^2 + (y - z)^2 &\leq 2 \wedge 2x^2 + (y + z)^2 \leq 2 \wedge \\ 2y^2 + (x - z)^2 &\leq 2 \wedge 2y^2 + (x + z)^2 \leq 2 \wedge \\ (x - y)^2 + 2z^2 &\leq 2 \wedge (x + y)^2 + 2z^2 \leq 2 \end{aligned}$$

(2) Sphericon

$$\begin{aligned} (x^2 + y^2 \leq (|z| - 1)^2 \wedge x \geq 0 \wedge -1 \leq z \leq 1) \vee \\ (x^2 + z^2 \leq (|y| - 1)^2 \wedge x \leq 0 \wedge -1 \leq y \leq 1) \end{aligned}$$

(3) Steinmetz 4-solid

$$\begin{aligned} x^2 + y^2 &\leq 1 \wedge 9x^2 + y^2 + 8z^2 \leq 9 + 164/29yz \wedge \\ 3x^2 + 284/41xy + 7y^2 + 82/29yz + 8z^2 &\leq 9 + 49/10xz \wedge \\ 3x^2 + 7y^2 + 49/10xz + 82/29yz + 8z^2 &\leq 9 + 284/41xy \end{aligned}$$

TABLE 3. Intersections of solids with $9(x + y + z)^2 > z^2 + 1$.

| <i>Ex #</i> | <i>dim</i> | <i>t_{CAD}</i> | <i>t_{SP}</i> | <i>t_{ADJ}</i> | <i>N_{CELL}</i> | <i>N_{ADJ}</i> | <i>N_{CC}</i> |
|-------------|------------|------------------------|-----------------------|------------------------|-------------------------|------------------------|-----------------------|
| 1 | 3 | 254 | 614 | 189 | 156688 | 4320078 | 2 |
| 2 | 3 | 59.5 | 82.6 | 27.0 | 54256 | 767462 | 2 |
| 3 | 3 | 52.1 | 78.2 | 20.1 | 24476 | 461614 | 2 |
| 4 | 3 | 8.11 | 5.42 | 3.06 | 17152 | 84162 | 2 |
| 5 | 3 | 47.9 | 53.5 | 15.1 | 11756 | 349976 | 2 |

(4) *Solid capsule*

$$x^2 + y^2 + (z - 1/2)^2 \leq 1 \vee x^2 + y^2 + (z + 1/2)^2 \leq 1 \vee \\ -1/2 \leq z \leq 1/2 \wedge x^2 + y^2 \leq 1$$

(5) *Reuleaux tetrahedron*

$$x^2 + y^2 + (19/31 + z)^2 \leq 1 \wedge \\ (x - 15/26)^2 + y^2 + (z - 9/44)^2 \leq 1 \wedge \\ (x + 11/38)^2 + (y - 1/2)^2 + (z - 9/44)^2 \leq 1 \wedge \\ (x + 11/38)^2 + (y + 1/2)^2 + (z - 9/44)^2 \leq 1$$

The details are given in Table 3.

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